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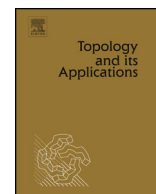


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Hom complexes and hypergraph colorings[☆]



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ABSTRACT

Babson and Kozlov (2006) [2] studied Hom-complexes of graphs with a focus on graph colorings. In this paper, we generalize Hom-complexes to r -uniform hypergraphs (with multiplicities) and study them mainly in connection with hypergraph colorings. We reinterpret a result of Alon, Frankl and Lovász (1986) [1] by Hom-complexes and show a hierarchy of known lower bounds for the chromatic numbers of r -uniform hypergraphs (with multiplicities) using Hom-complexes.

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1. Introduction

1.1. Hom-complexes of graphs

Since Lovász solved the famous Kneser conjecture by relating the chromatic number of a given graph to connectivity of its neighborhood complex [7], it is a standard method to study combinatorial properties of graphs by relating them with topological properties of appropriately constructed polyhedral complexes from graphs. Then as is seen in [5], a plenty of complexes have been constructed from graphs. Among others, let us consider Hom-complexes which were first introduced by Lovász and studied further by Babson and Kozlov [2,10,11]. Compared to other complexes of graphs, the construction of Hom-complex $\text{Hom}(G, H)$ for graphs G, H is quite natural; it is a space of maps from G to H . Moreover, some complexes of graphs concerning colorings are realized by special Hom-complexes [2,10] by which one can easily understand related construction. For example, a result of Lovász [7] can be reproved easily by using Hom-complexes as follows.

Let us start with a standard observation. Recall that an n -coloring of a graph G is a labeling of vertices of G by n colors in such a way that adjacent vertices have distinct colors. Then if K_n denotes the complete graph with n vertices, there is a one-to-one correspondence between n -colorings of G and homomorphisms of G into K_n . Suppose G admits an n -coloring. Then since the Hom-complex $\text{Hom}(G, H)$ is natural with respect to G, H , there is a map

$$\text{Hom}(T, G) \rightarrow \text{Hom}(T, K_n) \quad (1)$$

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for any graph T . Specialize T to the complete graph K_2 with 2 vertices. Then a natural C_2 -action on K_2 yields C_2 -actions on both $\text{Hom}(T, G)$ and $\text{Hom}(T, K_n)$, and furthermore, the map (1) is a C_2 -map for $T = K_2$, where C_k denotes the cyclic group of order k . One can easily see that the C_2 -actions are free and can also easily count the dimension of $\text{Hom}(K_2, K_n)$ as $n - 2$ by definition. Then it follows from the Borsuk–Ulam theorem that

$$\text{conn Hom}(K_2, G) \leq n - 3,$$

where $\text{conn } X$ denotes connectivity of a space X . Finally, since $\text{Hom}(K_2, G)$ has the homotopy type of the neighborhood complex of G as in [2], we obtain the result of Lovász [7]. The point of this proof is that we can get C_2 -actions and a C_2 -map quite naturally, which is often the most difficult part of the above mentioned topological method for graphs.

1.2. Generalization to r -graphs

Let us now generalize graphs to r -uniform hypergraphs. Recall that an r -uniform hypergraph (or an r -graph, for short) G consists of the vertex set and the edge set which is a collection of r elements subsets of the vertex set. Then 2-graphs are simple graphs, for instance. Homomorphisms of r -graphs are obviously defined. In [11, Definition 9.24], Kozlov suggested a recipe to construct a space of a collection of maps between finite sets. Then one can define Hom -complexes for r -graphs as well. We would like to study colorings of r -graphs by using Hom -complexes as in the above case of graphs. Colorings of graphs are generalized to r -graphs as follows. An n -coloring of an r -graph is a labeling of vertices by n colors such that each edge contains more than 2 colors. Then for $r \geq 3$, colorings of r -graphs cannot be realized as homomorphisms. Then in order to study r -graph colorings by Hom -complexes, we must extend the category of r -graphs so that colorings become homomorphisms. If we extend the category of r -graphs to that of all hypergraphs, colorings become homomorphisms. However, this category is too big to control objects. So we need a much smaller extension of the category of r -graphs. For this purpose, we will consider r -graphs with multiplicities which were first introduced by Lange [6] in a different context. Then we will study colorings of r -graphs with multiplicities through Hom -complexes. More precisely, we will give a lower bound for the chromatic numbers of r -graphs with multiplicities using group actions on special Hom -complexes. Alon, Frankl and Lovász [1] defined certain simplicial complexes of r -graphs (without multiplicities) and gave a lower bound for the chromatic numbers by a rather tricky construction. We will show that these complexes are essentially the same as the above special Hom -complexes, and then we can interpret their construction in terms of Hom -complexes, which will make things clear. We will also consider Hom_+ -complexes of r -graphs with multiplicities (cf. [11, Definition 20.1]) and show a hierarchy among lower bounds for the chromatic numbers.

1.3. Organization

The organization of the paper is as follows. In Section 2, we introduce r -graphs with multiplicities generalizing r -graphs by which we can study colorings of r -graphs as special homomorphisms. In Section 3, we recall a general construction of Hom -complexes of classes of maps between finite sets and then apply it to r -graphs with multiplicities. We show analogy of results of Babson and Kozlov [2] for Hom -complexes of r -graphs with multiplicities and give a lower bound for the chromatic number by special Hom -complexes. In Section 4, we show that the box-edge complexes of Alon, Frankl and Lovász [1] are realized by the above special Hom -complexes, by which we see that the above lower bound is the same as the one given by Alon, Frankl and Lovász [1]. In Section 5, we consider Hom_+ -complexes of r -graphs with multiplicities and give another lower bound for the chromatic number. By comparing Hom -complexes and Hom_+ -complexes, we show a hierarchy among the above two lower bounds.

2. r -Graphs with multiplicities

2.1. r -Graphs

Let us explain in detail why we introduce r -graphs with multiplicities. Recall that an r -uniform hypergraph (r -graph, for short) G is a pair of a finite set $V(G)$ and a collection $E(G)$ of r elements subsets of $V(G)$. $V(G)$ and $E(G)$ are respectively called the vertex set and the edge set of G . For r -graphs G, H , a homomorphism $f: G \rightarrow H$ is a map $f: V(G) \rightarrow V(H)$ satisfying $f_*(E(G)) \subset E(H)$. Our objects are colorings of r -graphs. An n -coloring of an r -graph G is a map $c: V(G) \rightarrow [n]$ such that if $\{v_1, \dots, v_r\} \in E(G)$, $\{c(v_1), \dots, c(v_r)\} \subset [n]$ is not a singleton, where $[n] = \{1, 2, \dots, n\}$. Then one sees that colorings cannot be realized by homomorphisms in general as in the case of graphs. Then generalizing r -graphs, we introduce r -graphs with multiplicities among which colorings are homomorphisms.

2.2. r -Graphs with multiplicities

Recall that the n th symmetric product of a set V is defined as

$$\text{SP}^n(V) = \underbrace{V \times \cdots \times V}_n / \Sigma_n,$$

where the action of the symmetric group Σ_n is given as $\sigma(v_1, \dots, v_n) = (v_{\sigma(1)}, \dots, v_{\sigma(n)})$ for $\sigma \in \Sigma_n$ and $v_1, \dots, v_n \in V$. We denote an element of $\text{SP}^r(V)$ by $v_1 \cdots v_n$ for $v_1, \dots, v_n \in V$. In [6], Lange used multisets which are naturally identified with elements of symmetric products. We now define r -graphs with multiplicities and their homomorphisms and colorings.

Definition 1.

- (1) An r -graph with multiplicities G consists of a finite set $V(G)$ and a subset $E(G)$ of $\text{SP}^r(V(G)) \setminus \Delta$, where $\Delta = \{v \cdots v \in \text{SP}^r(V(G))\}$. $V(G)$ and $E(G)$ are called the vertex set and the edge set of G , respectively.
- (2) Let G, H be r -graphs with multiplicities. A homomorphism $f: G \rightarrow H$ is a map $g: V(G) \rightarrow V(H)$ satisfying $f_*(E(G)) \subset E(H)$.
- (3) An n -coloring of an r -graph with multiplicities G is a map $c: V(G) \rightarrow [n]$ such that if $v_1 \cdots v_r \in E(G)$, $\{c(v_1), \dots, c(v_r)\} \subset [n]$ is not a singleton.
- (4) The chromatic number $\chi(G)$ of an r -graph with multiplicities G is the minimum integer n such that G admits an n -coloring.

Remark 2. If we allow r -graphs with multiplicities to have diagonal edges in Δ , some r -graphs do not admit any colorings. Since we will study colorings, we have omitted diagonal edges from r -graphs with multiplicities.

Since r elements subsets of a set V may be regarded as elements of $\text{SP}^r(V) \setminus \Delta$, r -graphs with multiplicities and their homomorphisms and colorings include r -graphs and their homomorphisms and colorings.

Define an r -graph with multiplicities $\mathcal{K}_n^{(r)}$ as the maximum r -graph with multiplicities with n vertices. Namely,

$$V(\mathcal{K}_n^{(r)}) = [n] \quad \text{and} \quad E(\mathcal{K}_n^{(r)}) = \text{SP}^r([n]) \setminus \Delta.$$

It is clear that there is the desired property as follows.

Proposition 3. There is a one-to-one correspondence between n -colorings of an r -graph with multiplicities G and homomorphisms from G to $\mathcal{K}_n^{(r)}$.

3. Hom-complexes of r -graphs with multiplicities

3.1. General Hom-complexes

Let us first recall a recipe of general Hom-complexes suggested by Kozlov [11, Definition 9.24]. Let S, T be finite sets. Then a map $S \rightarrow T$ is identified with an element of T^S . Let Δ^T be the simplex whose vertex set is T . Since T^S is the vertex set of a direct product $\prod_S \Delta^T$, a map $S \rightarrow T$ is identified with a vertex of $\prod_S \Delta^T$. This simple observation leads us to the following definition of Hom-complexes which may be regarded as spaces of given maps between finite sets.

Definition 4. Let S, T be finite sets and \mathcal{C} be a class of maps from S to T . The Hom-complex $\text{Hom}^{\mathcal{C}}(S, T)$ is the maximum subcomplex of $\prod_S \Delta^T$ whose vertex set is \mathcal{C} .

Let S, T be finite sets and \mathcal{C} be a class of maps from S to T . Given a map $f: T \rightarrow T'$ with T' finite and a class \mathcal{D} of maps from S to T' . If $f_*(\mathcal{C}) \subset \mathcal{D}$, we can define a map of polyhedral complexes

$$f_*: \text{Hom}^{\mathcal{C}}(S, T) \rightarrow \text{Hom}^{\mathcal{D}}(S, T')$$

by sending $h \in \mathcal{C}$ to $f \circ h \in \mathcal{D}$. Dually, given a map $g: S' \rightarrow S$ and a class \mathcal{E} of maps from S' to T satisfying $g^*(\mathcal{C}) \subset \mathcal{E}$, we can also define a map of polyhedral complexes

$$g^*: \text{Hom}^{\mathcal{C}}(S, T) \rightarrow \text{Hom}^{\mathcal{E}}(S', T)$$

by sending $h \in \mathcal{E}$ to $h \circ g \in \mathcal{C}$.

By definition of the above induced maps, we have the following functoriality.

Proposition 5. Let S, T be finite sets and \mathcal{C} be a class of maps from S to T .

- (1) Let T_1, T_2 be finite sets and $\mathcal{D}_1, \mathcal{D}_2$ be classes of maps from S to T_1 and T_2 , respectively. If maps $f_1: T \rightarrow T_1$ and $f_2: T_1 \rightarrow T_2$ satisfy $(f_1)_*(\mathcal{C}) \subset \mathcal{D}_1$ and $(f_2)_*(\mathcal{D}_1) \subset \mathcal{D}_2$, the induced maps on Hom-complexes satisfy

$$(f_2 \circ f_1)_* = (f_2)_* \circ (f_1)_*.$$

(2) Let S_1, S_2 be finite sets and $\mathcal{E}_1, \mathcal{E}_2$ be classes of maps from S_1 and S_2 to T , respectively. If maps $g_1 : S_1 \rightarrow S$ and $g_2 : S_2 \rightarrow S_1$ satisfy $(g_1)^*(C) \subset \mathcal{E}_1$ and $(g_2)^*(\mathcal{E}_1) \subset \mathcal{E}_2$, the induced maps on Hom -complexes satisfy

$$(g_2 \circ g_1)^* = (g_1)^* \circ (g_2)^*.$$

3.2. Hom -complexes of r -graphs with multiplicities

Let us return to r -graphs with multiplicities. Homomorphisms of r -graphs with multiplicities are maps between vertices satisfying certain properties. Since we are assuming vertex sets of r -graphs with multiplicities to be finite, we can apply the above general construction of Hom -complexes to r -graphs with multiplicities.

Definition 6. Let G, H be r -graphs with multiplicities and \mathcal{C} be the set of homomorphisms from G to H . The Hom -complex $\text{Hom}(G, H)$ is defined as $\text{Hom}^{\mathcal{C}}(V(G), V(H))$.

By Proposition 5, we have the following.

Proposition 7. Let $\mathbf{Graph}^{(r)}$ and \mathbf{Poly} be the categories of r -graphs with multiplicities and polyhedral complexes, respectively. Then

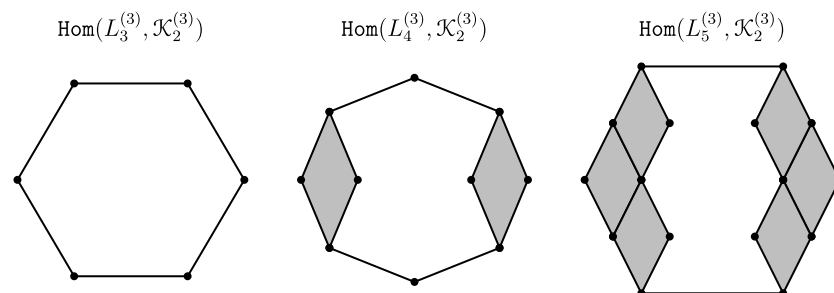
$$(\mathbf{Graph}^{(r)})^{\text{op}} \times \mathbf{Graph}^{(r)} \rightarrow \mathbf{Poly}, \quad (G, H) \mapsto \text{Hom}(G, H)$$

is a functor.

By Proposition 3, the Hom -complex $\text{Hom}(G, \mathcal{K}_n^{(r)})$ for an r -graph with multiplicities G is considered as a space of n -colorings of G . Then $\text{Hom}(G, \mathcal{K}_n^{(r)})$ is especially important, and hence we here give some easy examples. Let $L_n^{(r)}$ denote the line r -graph with n vertices. Namely, $L_n^{(r)}$ is defined as

$$V(L_n^{(r)}) = [n] \quad \text{and} \quad E(L_n^{(r)}) = \{i, i+1, \dots, i+r-1 \mid i=1, \dots, n-r+1\}.$$

Then $\text{Hom}(L_n^{(3)}, \mathcal{K}_2^{(3)})$ for $n=3, 4, 5$ are given as follows.



Note that $\text{Hom}(L_n^{(3)}, \mathcal{K}_2^{(3)})$ for $n=3, 4, 5$ have the same homotopy type. This will be justified below in a more general setting.

Let $C_n^{(r)}$ be the cyclic r -graph with n vertices. That is, $C_n^{(r)}$ is given as

$$V(C_n^{(r)}) = \mathbb{Z}/n \quad \text{and} \quad E(C_n^{(r)}) = \{i, i+1, \dots, i+r-1 \mid i \in \mathbb{Z}/n\}.$$

Let us next consider $\text{Hom}(C_n^{(3)}, \mathcal{K}_2^{(3)})$. Since $C_3^{(3)} = L_3^{(3)}$, $\text{Hom}(C_3^{(3)}, \mathcal{K}_2^{(3)})$ is a hexagon. One can easily see that $\text{Hom}(C_4^{(3)}, \mathcal{K}_2^{(3)})$ consists of discrete six points and that $\text{Hom}(C_5^{(3)}, \mathcal{K}_2^{(3)})$ is the outer polygon of $\text{Hom}(L_5^{(3)}, \mathcal{K}_2^{(3)})$. Then their homotopy types are not the same.

3.3. Lower bound for the chromatic number

By functoriality of $\text{Hom}(G, H)$, group actions on G and H induce those on $\text{Hom}(G, H)$. We next consider these group actions for special G . Let $K_n^{(r)}$ be the maximum r -graph with n vertices. Namely,

$$V(K_n^{(r)}) = [n] \quad \text{and} \quad E(K_n^{(r)}) = \text{SP}^r([n]) \setminus \Delta_n,$$

where $\Delta_n = \{v_1 \cdots v_r \in \text{SP}^r([n]) \mid v_i \neq v_j \text{ for } i \neq j\}$. Notice that by a cyclic permutation of vertices, the cyclic group C_n acts on $K_n^{(r)}$.

Lemma 8. If r is a prime, the induced C_r -action on $\text{Hom}(K_r^{(r)}, G)$ is free.

Proof. Any face of $\text{Hom}(K_r^{(r)}, G)$ is of the form $\Delta^{S_1} \times \cdots \times \Delta^{S_r}$ such that each $(v_1, \dots, v_r) \in S_1 \times \cdots \times S_r$ satisfies $v_1 \cdots v_r \in E(G)$. Then in particular, there is no diagonal element (v, \dots, v) in $S_1 \times \cdots \times S_r$. Let g be a non-trivial element of C_r . Then by renumbering if necessary, we have

$$g \cdot (v_1, \dots, v_r) = (v_r, v_1, \dots, v_{r-1})$$

for $(v_1, \dots, v_r) \in S_1 \times \cdots \times S_r$ since r is a prime. Suppose $g(\Delta^{S_1} \times \cdots \times \Delta^{S_r}) = \Delta^{S_1} \times \cdots \times \Delta^{S_r}$. Then we have observed that elements of S_1 belong to all S_1, \dots, S_r , a contradiction. \square

Using the index of the above free group action [8, Definition 6.2.3], we give a lower bound for the chromatic numbers of r -graphs with multiplicities. We set some notation. Let Γ be a non-trivial finite group (with the discrete topology). Let $E_n \Gamma$ be the join of $n+1$ copies of Γ on which Γ acts diagonally. Then this Γ -action on $E_n \Gamma$ is free and $E_n \Gamma$ has the homotopy type of a wedge of n -dimensional spheres. For a free Γ -complex X , the Γ -index of X is defined as

$$\text{ind}_\Gamma X = \min\{n \mid \text{there is a } \Gamma\text{-map } X \rightarrow E_n \Gamma\}.$$

Let us list basic properties of $\text{ind}_\Gamma X$.

Proposition 9. Let Γ be a non-trivial finite group and let X, Y be free Γ -complexes.

(1) If there is a Γ -map from X to Y , we have

$$\text{ind}_\Gamma X \leq \text{ind}_\Gamma Y.$$

(2) The join $X * Y$ is a free Γ -space by the diagonal Γ -action for which it holds that

$$\text{ind}_\Gamma (X * Y) \leq \text{ind}_\Gamma X + \text{ind}_\Gamma Y + 1.$$

(3) It holds that

$$\text{conn } X + 1 \leq \text{ind}_\Gamma X \leq \dim X.$$

Proof. (1) follows from definition and (2) follows from the fact that $E_n \Gamma = E_m \Gamma * E_{n-m-1} \Gamma$. By the Borsuk–Ulam theorem due to Dold [3], we have $\text{ind}_\Gamma E_n \Gamma = n$. Then (3) is shown by an easy obstruction argument. \square

Put $B_n \Gamma = E_n \Gamma / \Gamma$, $B \Gamma = \bigcup_{n \geq 1} B_n \Gamma$ and $E \Gamma = \bigcup_{n \geq 1} E_n \Gamma$. Then the natural projection $E \Gamma \rightarrow B \Gamma$ is the well-known Milnor's universal principal Γ -bundle. Let $\varphi: X/\Gamma \rightarrow B \Gamma$ be the classifying map of a free Γ -complex X . Then it follows that $\text{ind}_\Gamma X$ coincides with the minimum integer n such that φ factors through the inclusion $B_n \Gamma \rightarrow B \Gamma$, up to homotopy. By [4, Proposition 8.4], we obtain that $\text{ind}_\Gamma X$ is equal to the LS-category of the classifying map φ , implying that there are a lot of quantities estimating $\text{ind}_\Gamma X$ other than connectivity and dimension.

We now give a lower bound for the chromatic number of r -graphs with multiplicities.

Theorem 10. Let G be an r -graph with multiplicities. If r is a prime, there holds

$$\chi(G) \geq \frac{\text{ind}_{C_r} \text{Hom}(K_r^{(r)}, G) + 1}{r - 1} + 1.$$

Proof. By Lemma 8, $\text{Hom}(K_r^{(r)}, H)$ is a free C_r -complex for any r -graph with multiplicities H . Suppose there is an n -coloring of G , or equivalently, a homomorphism $f: G \rightarrow \mathcal{K}_n^{(r)}$. By Proposition 7, the induced map $f_*: \text{Hom}(K_r^{(r)}, G) \rightarrow \text{Hom}(K_r^{(r)}, \mathcal{K}_n^{(r)})$ is a C_r -map, implying

$$\text{ind}_{C_r} \text{Hom}(K_r^{(r)}, G) \leq \dim \text{Hom}(K_r^{(r)}, \mathcal{K}_n^{(r)})$$

by Proposition 9. We then count the dimension of $\text{Hom}(K_r^{(r)}, \mathcal{K}_n^{(r)})$. Any face of $\text{Hom}(K_r^{(r)}, \mathcal{K}_n^{(r)})$ is given as $\Delta^{S_1} \times \cdots \times \Delta^{S_r}$ such that $S_1, \dots, S_r \subset [n]$ and $S_1 \cap \cdots \cap S_r = \emptyset$. The maximum of $|S_1| + \cdots + |S_r|$ is $nr - n$ and then the dimension of $\text{Hom}(K_r^{(r)}, \mathcal{K}_n^{(r)})$ is $nr - n - r = (r - 1)(n - 1) - 1$, completing the proof. \square

By Proposition 9, we obtain the following.

Corollary 11. Let G be an r -graph with multiplicities. If r is a prime, we have

$$\chi(G) \geq \frac{\text{conn} \text{Hom}(K_r^{(r)}, G) + 2}{r - 1} + 1.$$

3.4. Homotopy lemmas

Let us recall three lemmas from [2, Proposition 3.2] and [11, Theorems 15.24, 15.28] which will be used below. We first set some notation. Let P be a poset. We denote the order complex of P by $\Delta(P)$. That is, $\Delta(P)$ is a simplicial complex whose n -simplices are chains in P of length $n + 1$. For $p \in P$, let

$$P_{\leq p} = \{q \in P \mid q \leq p\} \quad \text{and} \quad P_{\geq p} = \{q \in P \mid q \geq p\}.$$

We first state the famous Quillen fiber lemma.

Lemma 12. ([11, Theorem 15.28]) *Let $\varphi : P \rightarrow Q$ be a poset map between finite posets. If $\Delta(\varphi^{-1}(Q_{\leq q}))$ is contractible for any $q \in Q$, then $\Delta(\varphi) : \Delta(P) \rightarrow \Delta(Q)$ is a homotopy equivalence.*

We next state a variant of the Quillen fiber lemma proved in [2, Proposition 3.2].

Lemma 13. ([2, Proposition 3.2]) *For a poset map $\varphi : P \rightarrow Q$ between finite posets, suppose the following conditions.*

- (1) $\Delta(\varphi^{-1}(q))$ is contractible for any $q \in Q$.
- (2) For any $q \in Q$ and $p \in \varphi^{-1}(Q_{\geq q})$, the poset $\varphi^{-1}(q) \cap P_{\leq p}$ has the maximum.

Then $\Delta(\varphi) : \Delta(P) \rightarrow \Delta(Q)$ is a homotopy equivalence.

Finally, we recall the generalized nerve lemma which is frequently used in combinatorial algebraic topology. Let \mathcal{A} be a covering of a space X by non-empty subspaces A_1, \dots, A_n . Then we associate to \mathcal{A} a poset whose elements are non-empty intersections of A_1, \dots, A_n and the order is defined by inclusions. The nerve of \mathcal{A} is by definition the order complex of this poset associated to \mathcal{A} .

Lemma 14. ([11, Theorem 15.24]) *Let \mathcal{A} be a covering of a polyhedral complex K by non-empty subcomplexes A_1, \dots, A_n . Suppose that for any $i_1 < \dots < i_t$, there exists k such that $A_{i_1} \cap \dots \cap A_{i_t}$ is either empty or $(k - t + 1)$ -connected. Then K is k -connected if and only if so is the nerve of \mathcal{A} .*

3.5. Homotopy type of $\text{Hom}(K_m^{(r)}, \mathcal{K}_n^{(r)})$

As is mentioned above, for an r -graph with multiplicities G , the Hom -complex $\text{Hom}(G, \mathcal{K}_n^{(r)})$ is especially important. Then we determine the homotopy type of this Hom -complex in the special case $G = K_m^{(r)}$.

For $\mathbf{t} = (t_1, \dots, t_n)$ with non-negative integers t_1, \dots, t_n , we define a polyhedral complex $\Delta^m(\mathbf{t})$ as the subcomplex of $\underbrace{\Delta^{[n]} \times \dots \times \Delta^{[n]}}_m$ whose faces are $\Delta^{s_1} \times \dots \times \Delta^{s_m}$ such that $|\{k \in [m] \mid i \in S_k\}| \leq t_i$ for $i = 1, \dots, n$. Note that if $\mathbf{t}' = (t'_1, \dots, t'_n)$ satisfies $t_k, t'_k \geq m$ for some k and $t_i = t'_i$ for $i \neq k$, then $\Delta^m(\mathbf{t}) = \Delta^m(\mathbf{t}')$. As in the proof of Theorem 10, for $\mathbf{s} = (r - 1, \dots, r - 1) \in [m]^n$, we have

$$\Delta^m(\mathbf{s}) = \text{Hom}(K_m^{(r)}, \mathcal{K}_n^{(r)}).$$

We determine the homotopy type of $\Delta^m(\mathbf{t})$ and, consequently, the homotopy type of $\text{Hom}(K_m^{(r)}, \mathcal{K}_n^{(r)})$. For a polyhedral complex K , let $\mathcal{F}(K)$ denote the face poset of K .

Theorem 15. *For $\mathbf{t} = (t_1, \dots, t_n)$ with $0 \leq t_i \leq m$, $\Delta^m(\mathbf{t})$ has the homotopy type of a wedge of $(t_1 + \dots + t_n - m)$ -dimensional spheres.*

Proof. Put $|\mathbf{t}| = t_1 + \dots + t_n$. As in the proof of Theorem 10, one can easily deduce that the dimension of $\Delta^m(\mathbf{t})$ is $|\mathbf{t}| - m$. Then we only have to show that $\Delta^m(\mathbf{t})$ is $(|\mathbf{t}| - m - 1)$ -connected.

For $F \subset [n]$, put $\mathbf{t} - F = (t'_1, \dots, t'_n)$ such that

$$t'_i = \begin{cases} \max\{t_i - 1, 0\} & i \in F, \\ t_i & i \notin F. \end{cases}$$

Then if $F \subset F' \subset [n]$, we have $\Delta^\ell(\mathbf{t} - F) \supset \Delta^\ell(\mathbf{t} - F')$. We also have that if $|\mathbf{t}| - |F| - \ell < 0$ for $F \subset [n]$, $\Delta^\ell(\mathbf{t} - F) = \emptyset$. We now define a functor

$$\rho : \mathcal{F}(\text{sk}_{|\mathbf{t}| - m} \Delta^{[n]})^{\text{op}} \rightarrow \mathbf{Poly}$$

by $\rho(F) = \Delta^{m-1}(\mathbf{t} - F)$ and inclusions $\Delta^{m-1}(\mathbf{t} - F) \supset \Delta^{m-1}(\mathbf{t} - F')$ for $F \subset F' \in \mathcal{F}(\text{sk}_{|\mathbf{t}|-m}\Delta^{[n]})$, where $\text{sk}_k K$ denotes the k -skeleton of a polyhedral complex K . By definition, $\Delta^m(\mathbf{t})$ is the union of $\Delta^F \times \Delta^{m-1}(\mathbf{t} - F)$ for all non-empty $F \in \mathcal{F}(\text{sk}_{|\mathbf{t}|-m}\Delta^{[n]})$. Namely, we have

$$\Delta^m(\mathbf{t}) = \text{hocolim } \rho.$$

Since ρ maps every arrow to a cofibration, we get a homotopy equivalence

$$\text{hocolim } \rho \xrightarrow{\sim} \text{colim } \rho.$$

See [11, Theorem 15.19]. Notice that $\text{colim } \rho$ is covered by subcomplexes $\Delta^{m-1}(\mathbf{t} - \{i\})$ for $i \in [n]$ and that $\Delta^{m-1}(\mathbf{t} - F) \cap \Delta^{m-1}(\mathbf{t} - F') = \Delta^{m-1}(\mathbf{t} - F \cup F')$ for $F, F' \in \mathcal{F}(\text{sk}_{|\mathbf{t}|-m}\Delta^{[n]})$.

If $|\mathbf{t}| = m$, $\Delta^m(\mathbf{t})$ is a discrete finite set. Apply Lemma 14 to the above covering of $\text{colim } \rho$ inductively on $|\mathbf{t}| - m$. Thus we obtain the desired result. \square

Corollary 16. $\text{Hom}(K_m^{(r)}, \mathcal{K}_n^{(r)})$ has the homotopy type of a wedge of $((r-1)n - m)$ -dimensional spheres.

3.6. Vertex deletion and Hom -complexes

In [2, Proposition 5.1], a relation between vertex deletion of G and the homotopy type of $\text{Hom}(G, H)$ is considered when G, H are graphs. We prove analogy for r -graphs with multiplicities here by a quite similar way. In [2, Proposition 5.1], a condition for vertex deletion is given by a neighborhood of a vertex. As for graphs, a neighborhood of a vertex v is considered as both the set of vertices adjacent to v and the set of edges with the end v . As for r -graphs with multiplicities, these two sets cannot be identified for $r \geq 3$, and then we define two kinds of neighborhoods of vertices.

Let G be an r -graph with multiplicities. For a vertex v of G , we define $N(v)$ as the set of $v_1 \cdots v_s \in \text{SP}^s(V(G))$ for some $1 \leq s \leq r-1$ satisfying $\underbrace{v \cdots v}_{r-s} v_1 \cdots v_s \in E(G)$ and $v_1, \dots, v_s \neq v$. For $v_1 \cdots v_s \in \text{SP}^s(V(G))$ with $1 \leq s \leq r-1$, we also define

$\check{N}(v_1 \cdots v_s)$ as the set of vertices w of G satisfying $\underbrace{w \cdots w}_{r-s} v_1 \cdots v_s \in E(G)$.

For a vertex v of G , let $G \setminus v$ denote the maximum r -subgraph with multiplicities of G whose vertex set is $V(G) \setminus v$. We now state our result.

Theorem 17. Let G, H be an r -graph with multiplicities. Suppose that there are vertices u, v of G satisfying $N(u) \supset N(v)$. Then the inclusion $i: G \setminus v \rightarrow G$ induces a homotopy equivalence

$$i^*: \text{Hom}(G, H) \xrightarrow{\sim} \text{Hom}(G \setminus v, H).$$

Proof. As is mentioned above, the proof is quite analogous to [2, Proposition 5.1]. Note that any face of $\text{Hom}(T, H)$ for an r -graph with multiplicities T is identified with a map $V(T) \rightarrow 2^{V(H)} \setminus \emptyset$. For $\eta \in \mathcal{F}(\text{Hom}(G \setminus v, H))$, the fiber $\mathcal{F}(i^*)^{-1}(\eta)$ is the set of $\tau \in \mathcal{F}(\text{Hom}(G, H))$ satisfying

$$\tau|_{V(G) \setminus v} = \eta.$$

Since

$$\bigcap_{v_1 \cdots v_s \in N(v)} \bigcap_{(w_1, \dots, w_s) \in \eta(v_1) \times \cdots \times \eta(v_s)} \check{N}(w_1 \cdots w_s) \supset \bigcap_{v_1 \cdots v_s \in N(u)} \bigcap_{(w_1, \dots, w_s) \in \eta(v_1) \times \cdots \times \eta(v_s)} \check{N}(w_1 \cdots w_s) \supset \eta(u) \neq \emptyset,$$

we can define $v \in \mathcal{F}(i^*)^{-1}(\eta)$ by

$$v(v) = \bigcap_{v_1 \cdots v_s \in N(v)} \bigcap_{(w_1, \dots, w_s) \in \eta(v_1) \times \cdots \times \eta(v_s)} \check{N}(w_1 \cdots w_s)$$

and $v|_{V(G) \setminus v} = \eta$. By definition, v is the maximum of $\mathcal{F}(i^*)^{-1}(\eta)$, and thus in particular, the order complex $\Delta(\mathcal{F}(i^*)^{-1}(\eta))$ is contractible.

Choose $\tau \in \mathcal{F}(\text{Hom}(G, H))$ and $\eta \in \mathcal{F}(\text{Hom}(G \setminus v, H))$ satisfying $\tau(w) \supset \eta(w)$ for $w \in V(G) \setminus v$. Observe that $\mathcal{F}(i^*)^{-1}(\eta) \cap \mathcal{F}(\text{Hom}(G, H))_{\leq \tau}$ consists of $\sigma \in \mathcal{F}(\text{Hom}(G, H))$ satisfying $\sigma(v) \subset \tau(v)$ and $\sigma|_{V(G) \setminus v} = \eta$. Then it has the maximum μ such that $\mu(v) = \tau(v)$ and $\mu|_{V(G) \setminus v} = \eta$. We have seen that Lemma 13 can be applied to $\mathcal{F}(i^*): \mathcal{F}(\text{Hom}(G, H)) \rightarrow \mathcal{F}(\text{Hom}(G \setminus v, H))$, which completes the proof. \square

We generalize the above observation on the homotopy type of $\text{Hom}(L_n^{(3)}, \mathcal{K}_2^{(3)})$.

Corollary 18. Let $L_n^{(r)}$ be the line r -graph with n vertices as above, and let G be an r -graph with multiplicities. Then for $n \geq r$, we have

$$\text{Hom}(L_n^{(r)}, G) \simeq \text{Hom}(L_r^{(r)}, G).$$

Proof. If $n > r$, we have $N(n) \subset N(n-r)$. Then by Theorem 17, it holds that $\text{Hom}(L_n^{(r)}, G) \simeq \text{Hom}(L_{n-1}^{(r)}, G)$. Thus the result follows by induction on n . \square

4. Relation between box-edge complexes and Hom-complexes

4.1. Box-edge complexes

Let G be an r -graph (without multiplicities). In [1], Alon, Frankl and Lovász introduced a simplicial complex $\text{B}_{\text{edge}}(G)$ with a C_r -action which we call the box-edge complex of G , where we follow the name and the notation of [9]. By an ad-hoc and tricky construction concerning $\text{B}_{\text{edge}}(G)$, they gave a lower bound for the chromatic number of G . We will show that this construction is realized by special Hom-complexes of r -graphs with multiplicities, by which we can reprove and interpret a result of Alon, Frankl and Lovász [1, Proposition 2.1] in a quite natural way.

Let $\pi : V^n \rightarrow \text{SP}^n(V)$ denote the projection for a set V . Originally, the box-edge complexes were defined only for r -graphs (without multiplicities). However, their definition can be applied to r -graphs with multiplicities straightforwardly.

Definition 19. Let G be an r -graph with multiplicities. The box-edge complex of G is an abstract simplicial complex defined as

$$\text{B}_{\text{edge}}(G) = \{F \subset V(G)^r \mid \pi(F) \subset E(G)\} \quad (2)$$

on which the cyclic group C_r acts as the restriction of the permutation action on $V(G)^r$.

Notice here that as is shown in [1, Proposition 2.1], if r is a prime, the C_r -action on $\text{B}_{\text{edge}}(G)$ is free.

4.2. Result of Alon, Frankl and Lovász

We prove that the box-edge complex $\text{B}_{\text{edge}}(G)$ is given by a special Hom-complex.

Theorem 20. For an r -graph with multiplicities G , there is a C_r -map

$$\text{B}_{\text{edge}}(G) \rightarrow \text{Hom}(K_r^{(r)}, G)$$

which is a homotopy equivalence. In particular, if r is a prime, it is a C_r -homotopy equivalence.

Proof. The face poset of $\text{Hom}(K_r^{(r)}, G)$ is given as

$$\mathcal{F}(\text{Hom}(K_r^{(r)}, G)) = \{F_1 \times \cdots \times F_r \mid F_1, \dots, F_r \subset V(G) \text{ and } \pi(F_1 \times \cdots \times F_r) \subset E(G)\}, \quad (3)$$

where the order is given by inclusions. Then as the face poset of $\text{B}_{\text{edge}}(G)$ is given in (2), we can define a map

$$\varphi : \mathcal{F}(\text{B}_{\text{edge}}(G)) \rightarrow \mathcal{F}(\text{Hom}(K_r^{(r)}, G)), \quad F \mapsto \pi_1(F) \times \cdots \times \pi_r(F),$$

where $\pi_i : V(G)^r \rightarrow V(G)$ is the i th projection. Then by definition, φ is a C_r -map and hence so is $\Delta(\varphi)$.

Take any $F_1 \times \cdots \times F_r \in \mathcal{F}(\text{Hom}(K_r^{(r)}, G))$. Then the poset $\varphi^{-1}(\text{Hom}(K_r^{(r)}, G)_{\leq F_1 \times \cdots \times F_r})$ has the maximum $F_1 \times \cdots \times F_r$, implying that $\Delta(\varphi^{-1}(\text{Hom}(K_r^{(r)}, G)_{\leq F_1 \times \cdots \times F_r}))$ is contractible. Thus by Lemma 12, $\Delta(\varphi)$ is a homotopy equivalence. The desired map is the composite

$$\text{B}_{\text{edge}}(G) \xrightarrow{\cong} \Delta(\mathcal{F}(\text{B}_{\text{edge}}(G))) \xrightarrow{\Delta(\varphi)} \Delta(\mathcal{F}(\text{Hom}(K_r^{(r)}, G))) \xrightarrow{\cong} \text{Hom}(K_r^{(r)}, G),$$

where the first and the last arrows are the natural homeomorphisms between polyhedral complexes and their barycentric subdivision. Therefore we have established the first assertion. Suppose r is a prime. Then the C_r -action on $\text{Hom}(K_r^{(r)}, G)$ is free by Lemma 8. Moreover, the C_r -action on $\text{B}_{\text{edge}}(G)$ is also free as is noted above. Thus the second assertion follows from the first one. \square

Remark 21. Recently, Thansri [14, Corollary 4.9] showed that $\text{B}_{\text{edge}}(G)$ and $\text{Hom}(K_r^{(r)}, G)$ has the same Σ_r -equivariant simple homotopy type for an r -graph (without multiplicities) G .

By [Corollary 11](#), we obtain a result of Alon, Frankl and Lovász [[1, Proposition 2.1](#)].

Corollary 22. *Let G be an r -graph with multiplicities. If r is a prime, we have*

$$\chi(G) \geq \frac{\text{conn } \mathcal{B}_{\text{edge}}(G) + 2}{r - 1} + 1.$$

Alon, Frankl and Lovász [[1, §3](#)] proved [Corollary 22](#) by constructing a map from $\mathcal{B}_{\text{edge}}(G)$ into a Euclidean space with a certain C_r -action, which seems quite ad-hoc and tricky. Using Hom -complexes, this construction will turn out to be the induced map between Hom -complexes from a given coloring.

Let $M_{r,n}(\mathbb{R})$ be the space of $r \times n$ real matrices. We let C_r act on $M_{r,n}(\mathbb{R})$ as the cyclic permutation of rows. Let Y be a subspace of $M_{r,n}(\mathbb{R})$ consisting of matrices (a_{ij}) satisfying

$$\sum_{i=1}^r a_{ik} = 0, \quad \sum_{j=1}^n a_{\ell j} = 0 \quad \text{and} \quad \sum_{i,j} a_{ij}^2 \neq 0$$

for $k = 1, \dots, n$ and $\ell = 1, \dots, r$. Then Y is also a C_r -subspace of $M_{r,n}(\mathbb{R})$. Let G be an r -graph with multiplicities which admits an n -coloring, say c . Alon, Frankl and Lovász [[1, §3](#)] defined a C_r -map

$$\bar{c}: \mathcal{B}_{\text{edge}}(G) \rightarrow M_{r,n}(\mathbb{R})$$

by sending a vertex (v_1, \dots, v_r) of $\mathcal{B}_{\text{edge}}(G)$ to a matrix $\sum_{i=1}^r (E_{i,c(i)} - E_{i,c(i)+1})$, where $E_{i,j}$ is the matrix whose (i, j) entry is 1 and other entries are 0 and $E_{i,n+1}$ means $E_{i,1}$. They showed that \bar{c} has its image in Y and applied a special generalization of the Borsuk–Ulam theorem to obtain [Corollary 22](#).

We now define a map $g: \text{Hom}(K_r^{(r)}, \mathcal{K}_n^{(r)}) \rightarrow M_{r,n}(\mathbb{R})$ by sending a vertex $(i_1, \dots, i_r) \in [n]^r$ to a matrix $\sum_{k=1}^r (E_{k,i_k} - E_{k,i_k+1})$. Then one can easily see that g is a C_r -map and has its image in Y . By definition, we have the following.

Proposition 23. *Let G be an r -graph with multiplicities which has an n -coloring c . Then there is a commutative diagram*

$$\begin{array}{ccc} \text{Hom}(K_r^{(r)}, G) & \xrightarrow{c_*} & \text{Hom}(K_r^{(r)}, \mathcal{K}_n^{(r)}) \\ \downarrow \simeq & & \downarrow g \\ \mathcal{B}_{\text{edge}}(G) & \xrightarrow{\bar{c}} & Y \end{array}$$

where the left vertical arrow is as in [Theorem 20](#).

We close this section by remarking that the complex of an r -graph (without multiplicities) introduced by Kříž [[12, §4](#)] is the barycentric subdivision of $\text{Hom}(K_r^{(r)}, G)$ and then essentially the same as the box-edge complex $\mathcal{B}_{\text{edge}}(G)$.

5. Hom_+ -complexes and colorings

5.1. General Hom_+ -complexes

In [[11, Definition 20.1](#)], Hom_+ -complexes of graphs were introduced which are variants of Hom -complexes. As in the case of Hom -complexes, we can give a general recipe for Hom_+ -complexes of partial maps between finite sets and will apply it to r -graphs with multiplicities.

Let S, T be finite sets. A partial map from S to T is a map from a non-empty subset of S into T . Then a partial map from S to T is identified with an element of

$$(T \cup \{\emptyset\})^S \setminus (\emptyset, \dots, \emptyset).$$

Let K, L be abstract simplicial complexes. Recall that the join $K * L$ is an abstract simplicial complex whose simplices are of the form (σ, τ) where $\sigma \in K$, $\tau \in L$ and either σ or τ is not empty. Then a partial map from S to T is identified with a vertex of the join $*_{s \in S} \Delta^T$. Analogously to Hom -complexes, we are led to the following definition.

Definition 24. Let S, T be finite sets and \mathcal{C} be a class of partial maps from S to T . The Hom_+ -complex $\text{Hom}_+^{\mathcal{C}}(S, T)$ is defined as the maximum subcomplex of $*_{s \in S} \Delta^T$ whose vertex set is \mathcal{C} .

Analogously to Hom -complexes, we can define induced maps between Hom_+ -complexes under certain conditions and see that these induced maps satisfy naturality corresponding to [Proposition 5](#).

5.2. Hom_+ -complexes of r -graphs with multiplicities

Let G, H be r -graphs with multiplicities. A partial homomorphism from G to H is a map from a subset V of $V(G)$ into $V(H)$ which is a homomorphism from the maximum r -subgraph with multiplicities of G whose vertex set is V into H . We now define Hom_+ -complexes of r -graphs with multiplicities.

Definition 25. Let G, H be r -graphs with multiplicities. The Hom_+ -complex $\text{Hom}_+(G, H)$ is defined as $\text{Hom}_+^{\mathcal{C}}(V(G), V(H))$ for the set \mathcal{C} of all partial homomorphisms from G to H .

Similarly to Proposition 7, we have the following.

Proposition 26. Let $\mathbf{Graph}^{(r)}$ and \mathbf{Poly} be the categories of r -graphs with multiplicities and polyhedral complexes, respectively. Then

$$(\mathbf{Graph}^{(r)})^{\text{op}} \times \mathbf{Graph}^{(r)} \rightarrow \mathbf{Poly}, \quad (G, H) \mapsto \text{Hom}_+(G, H)$$

is a functor.

Then as in the case of Hom -complexes, we can construct group actions on Hom_+ -complexes by those on r -graphs with multiplicities. For instance, the natural C_r -action on $K_r^{(r)}$ induces a C_r -action on $\text{Hom}_+(K_r^{(r)}, G)$ for an r -graph with multiplicities G . Analogously to Lemma 8, we can prove the following.

Lemma 27. Let G be an r -graph with multiplicities. If r is a prime, the C_r -action on $\text{Hom}_+(K_r^{(r)}, G)$ is free.

Using this C_r -action, we obtain a lower bound for the chromatic numbers.

Theorem 28. Let G be an r -graph with multiplicities. If r is a prime, it holds that

$$\chi(G) \geq \frac{\text{ind}_{C_r} \text{Hom}_+(K_r^{(r)}, G) + 1}{r - 1}.$$

Proof. Note that the dimension of $\text{Hom}_+(K_r^{(r)}, \mathcal{K}_n^{(r)})$ is $nr - n - 1 = (r - 1)n - 1$. Then the result follow quite similarly to Theorem 10. \square

Corollary 29. Let G be an r -graph with multiplicities. If r is a prime, we have

$$\chi(G) \geq \frac{\text{conn} \text{Hom}_+(K_r^{(r)}, G) + 2}{r - 1}.$$

In [6, p. 5], Lange defined a complex $B_0(G)$ for an r -graph with multiplicities and gave a lower bound for the chromatic number of G by using $B_0(G)$. By definition, $B_0(G)$ coincides with $\text{Hom}_+(K_r^{(r)}, G)$ and a lower bound in Theorem 28 is the same as the one given by Lange.

As in Section 3, let us consider the homotopy type of $\text{Hom}_+(K_m^{(r)}, \mathcal{K}_n^{(r)})$. In the case of Hom_+ -complexes, one can describe $\text{Hom}_+(K_m^{(r)}, \mathcal{K}_n^{(r)})$ explicitly by Sarkaria's formula [13, (2.2)] as follows.

Theorem 30. ([13, (2.2)]) We have

$$\text{Hom}_+(K_m^{(r)}, \mathcal{K}_n^{(r)}) \cong *_n \text{sk}_{r-2} \Delta^{[m]}.$$

In particular, $\text{Hom}_+(K_m^{(r)}, \mathcal{K}_n^{(r)})$ has the homotopy type of a wedge of $\binom{m-1}{r-1}^n$ copies of $((r - 1)n - 1)$ -dimensional spheres.

5.3. Hierarchy of lower bounds for the chromatic number

Let G be an r -graph with multiplicities. We have obtained so far two kinds of lower bounds for the chromatic number of G , one is given by $\text{Hom}(K_r^{(r)}, G)$ in Theorem 10 and the other is given by $\text{Hom}_+(K_r^{(r)}, G)$ in Theorem 28. We have also seen that these lower bounds are related to formerly known ones [1, Proposition 2.1], [6, Theorem 3]. We describe $\text{Hom}_+(K_r^{(r)}, G)$ by using $\text{Hom}(K_r^{(r)}, G)$ and then get an inequality between the above lower bounds.

Theorem 31. For an r -graph with multiplicities G , there is a C_r -map

$$\mathrm{Hom}_+(K_r^{(r)}, G) \rightarrow \partial \Delta^{[r]} * \mathrm{Hom}(K_r^{(r)}, G)$$

which is a homotopy equivalence, where C_r acts diagonally on $\partial \Delta^{[r]} * \mathrm{Hom}(K_r^{(r)}, G)$.

Proof. Let P, Q be finite posets. Recall that the join $P * Q$ is a poset whose underlying set is $P \sqcup Q$ and order is defined as $x < y$ if either $x, y \in P$ with $x < y$, $x, y \in Q$ with $x < y$ or $x \in P, y \in Q$. Then it follows that

$$\Delta(P * Q) = \Delta(P) * \Delta(Q).$$

Note that the face poset of $\mathrm{Hom}_+(K_r^{(r)}, G)$ is the disjoint union of $\mathcal{F}(\mathrm{Hom}(K_r^{(r)}, G))$ in (3) and

$$\left\{ F_1 \times \cdots \times F_r \mid F_1, \dots, F_r \subset V(G), F_i = \emptyset \text{ for some } i \text{ and } \bigcup_{i=1}^r F_i \neq \emptyset \right\},$$

where the order is given by inclusions and $F_1 \times \cdots \times F_n$ with $F_{i_1}, \dots, F_{i_k} \neq \emptyset$ and $F_j = \emptyset$ for $j \neq i_1, \dots, i_k$ means $F_{i_1} \times \cdots \times F_{i_k}$. We then define a poset map

$$\varphi : \mathcal{F}(\mathrm{Hom}_+(K_r^{(r)}, G)) \rightarrow \mathcal{F}(\partial \Delta^{[r]}) * \mathcal{F}(\mathrm{Hom}(K_r^{(r)}, G))$$

as

$$\varphi(F_1 \times \cdots \times F_r) = \begin{cases} \{i_1, \dots, i_k\} \in \mathcal{F}(\partial \Delta^{[r]}) & \bigcup_{i \neq i_1, \dots, i_k} F_i = \emptyset \text{ and } F_{i_1}, \dots, F_{i_k} \neq \emptyset, \\ F_1 \times \cdots \times F_r \in \mathcal{F}(\mathrm{Hom}(K_r^{(r)}, G)) & F_1, \dots, F_r \neq \emptyset. \end{cases}$$

By definition, φ is a C_r -map. For $F_1 \times \cdots \times F_r \in \mathcal{F}(\mathrm{Hom}(K_r^{(r)}, G)) \subset \mathcal{F}(\partial \Delta^{[r]}) * \mathcal{F}(\mathrm{Hom}(K_r^{(r)}, G))$, $\varphi^{-1}((\mathcal{F}(\partial \Delta^{[r]}) * \mathcal{F}(\mathrm{Hom}(K_r^{(r)}, G)))_{\leq F_1 \times \cdots \times F_r})$ has the maximum $F_1 \times \cdots \times F_r$. For $\{i_1, \dots, i_k\} \in \partial \Delta^{[r]} \subset \mathcal{F}(\partial \Delta^{[r]}) * \mathcal{F}(\mathrm{Hom}(K_r^{(r)}, G))$, $\varphi^{-1}((\mathcal{F}(\partial \Delta^{[r]}) * \mathcal{F}(\mathrm{Hom}(K_r^{(r)}, G)))_{\leq \{i_1, \dots, i_k\}})$ has the maximum

$$F_1 \times \cdots \times F_r, \quad F_{i_1} = \cdots = F_{i_k} = [n] \quad \text{and} \quad \bigcup_{i \neq i_1, \dots, i_k} F_i = \emptyset.$$

Then for any $x \in \mathcal{F}(\partial \Delta^{[r]}) * \mathcal{F}(\mathrm{Hom}(K_r^{(r)}, G))$, $\Delta((\mathcal{F}(\partial \Delta^{[r]}) * \mathcal{F}(\mathrm{Hom}(K_r^{(r)}, G)))_{\leq x})$ is contractible, and it follows from Lemma 12 that $\Delta(\varphi)$ is a homotopy equivalence. Thus the composite

$$\begin{aligned} \mathrm{Hom}_+(K_r^{(r)}, G) &\xrightarrow{\cong} \Delta(\mathcal{F}(\mathrm{Hom}_+(K_r^{(r)}, G))) \xrightarrow{\Delta(\varphi)} \Delta(\mathcal{F}(\partial \Delta^{[r]}) * \mathcal{F}(\mathrm{Hom}(K_r^{(r)}, G))) \\ &= \Delta(\mathcal{F}(\partial \Delta^{[r]}) * \mathcal{F}(\mathrm{Hom}(K_r^{(r)}, G))) \xrightarrow{\cong} \partial \Delta^{[r]} * \mathrm{Hom}(K_r^{(r)}, G) \end{aligned}$$

is the desired homotopy equivalence, where the first and the last arrows are the natural homeomorphisms. \square

Corollary 32. Let G be an r -graph with multiplicities. If r is a prime, there holds

$$\begin{aligned} \chi(G) &\geq \frac{\mathrm{ind}_{C_r} \mathrm{Hom}(K_r^{(r)}, G) + 1}{r - 1} + 1 \geq \frac{\mathrm{ind}_{C_r} \mathrm{Hom}_+(K_r^{(r)}, G) + 1}{r - 1} \geq \frac{\mathrm{ind}_{C_r} \mathrm{Hom}(K_r^{(r)}, G) + 1}{r - 1} \\ &\geq \frac{\mathrm{conn} \mathrm{Hom}(K_r^{(r)}, G) + 1}{r - 1} = \frac{\mathrm{conn} \mathrm{Hom}_+(K_r^{(r)}, G) + 1}{r - 1} - 1. \end{aligned}$$

Proof. The first inequality follows from Theorem 10 and the second from Proposition 9 and Theorem 31. As in the proof of Theorem 31, $\mathcal{F}(\mathrm{Hom}(K_r^{(r)}, G))$ is a subposet of $\mathcal{F}(\mathrm{Hom}_+(K_r^{(r)}, G))$ including the C_r -actions. Then there is a C_r -map $\mathrm{Hom}(K_r^{(r)}, G) \rightarrow \mathrm{Hom}_+(K_r^{(r)}, G)$, implying the third inequality by Proposition 9. The fourth inequality follows from Proposition 9 and the last equality from Theorem 31. \square

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